University of California, Berkeley
Physics 110A, Section 2, Spring 2003 (*Strovink*)

PROBLEM SET 1

1.
(a.)
Prove that
$$(AB)^t = B^t A^t \ ,$$
where the transpose
$$(A^t)_{ij} \equiv A_{ji} \ ,$$
and A and B are second-rank tensors.
(b.)
Show that the rule (Griffiths 1.32) for transforming a second-rank tensor T between mutually rotated coordinate systems can be written
$$\bar{T}_{ij} = R_{ik} T_{kl} (R^t)_{lj} \ ,$$
where R is the rotation matrix and (throughout this problem set) repeated indices are summed. In matrix notation this is equivalent to
$$\bar{T} = R T R^t \ .$$

2.
Prove that
$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \ ,$$
where the Levi-Civita density $\epsilon_{ijk} \equiv 1$ for $ijk =$ cyclical permutations of 123, $\equiv -1$ for cyclical permutations of 321, otherwise $\equiv 0$; and the Kronecker delta function $\delta_{ij} \equiv 1$ for $i = j$, otherwise $\equiv 0$.

[*Hint*: In general there are 81 different combinations of i, j, l, and m. However, by symmetry, it is sufficient to consider only the case $i = 1$. So, for all 27 possible combinations of j, l, and m, show that the left- and right-hand sides of the equation are equal.]

3.
Using the result of Problem (**2.**), and writing
$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m \ ,$$
prove the BAC-CAB rule (Griffiths 1.17):
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \ .$$

4.
Use the BAC-CAB rule to decompose any vector $\vec{F}$ into a part that is parallel to $\hat{n}$, plus a part that is perpendicular to $\hat{n}$, where $\hat{n}$ is an arbitrary unit vector.

5.
(a.)
If S is a symmetric tensor ($S_{ij} = S_{ji}$), and A is an antisymmetric tensor ($A_{ij} = -A_{ji}$), show that
$$S_{ij} A_{ij} = 0 \ .$$
(b.)
Writing
$$[(\nabla \times (\nabla f)]_i = \epsilon_{ijk} \partial_j \partial_k f \ ,$$
where
$$\partial_j \equiv \frac{\partial}{\partial x_j} \ ,$$
and using the fact that $\partial_j \partial_k f$ is symmetric under interchange of j and k, prove that *curl grad* $f = 0$ (Griffiths 1.44).
(c.)
Writing
$$\nabla \cdot (\nabla \times \vec{F}) = \epsilon_{ijk} \partial_i \partial_j F_k \ ,$$
prove that *div curl* $\vec{F} = 0$ (Griffiths 1.46).

6.
Consider the vector function $\vec{F}(\vec{r}) = \hat{\phi}$.
(a.)
Calculate
$$\oint_C \vec{F} \cdot d\vec{l} \, ,$$
where C is a circle of radius r in the xy plane centered at the origin, and $d\vec{l}$ is counterclockwise.

(b.)
Calculate

$$\int_H (\nabla \times \vec{F}) \cdot d\vec{a} \; ,$$

where H is the hemisphere above and bounded by curve C, and $d\vec{a}$ is outward.
(c.)
Calculate

$$\int_D (\nabla \times \vec{F}) \cdot d\vec{a} \; ,$$

where D is the disk bounded by curve C, and $d\vec{a}$ is upward.
(d.)
Verify that Stokes' theorem (Griffiths 1.57)

$$\oint \vec{F} \cdot d\vec{l} = \int (\nabla \times \vec{F}) \cdot d\vec{a}$$

holds for both surfaces (b.) and (c.).

7.
Using Dirac delta functions (Griffiths 1.96), express the following electric charge distributions as three-dimensional charge densities $\rho(\vec{r})$. [Check your answer by integrating $\rho(\vec{r})$ over the appropriate volume to obtain the total charge (or charge per unit length).]
(a.)
A charge Q, uniformly distributed over a disk of radius b in the xy plane ($z = 0$).
(b.)
A charge per unit length λ, uniformly distributed within an infinitely long, infinitesimally thin wire lying on the z axis.
(c.)
A charge per unit length λ, uniformly distributed over an infinitely long cylindrical surface of radius b centered on the z axis.

8.
Express the vector field

$$\vec{H}(x, y, z) = \hat{x}x^2y + \hat{y}y^2z + \hat{z}z^2x$$

as the sum of an irrotational field $\vec{F}$ and a solenoidal field $\vec{G}$ (Griffiths 1.105). [*Hint*: Use the fact that $\nabla \cdot \vec{H} = \nabla \cdot \vec{F}$ and that $\vec{F} = -\nabla V$, where V is a scalar potential. Obtain a second-order partial differential equation for V and guess a solution to it.]